Introduction

**Iterative Methods (Jacobi / Gauss-Siedel / SOR)**

One way to find the solution of is to use iterative methods, where an initial guess for a solution is made, and then iteratively refined.

Every matrix , where L is lower triangular, D is diagonal, U is upper triangular.

Jacobi’s Method

One iterative method for finding the root of is the **Jacabi’s Method**, where you let

Then you can define a sequence recursively by

Thinking in terms of equations instead of matrices like earlier mathematicians, we have a system and an initial guess . To get the first component of our next vector, we treat the rest of the components of our previous vector as correct, and use the first equation to solve for that first component:

Similarly, to get the second component of your next vector:

Gauss-Siedel

Another iterative method for finding the root of is the **Gauss-Siedel Method**

Choose

Then our sequence can be iteratively defined by

is an equation that can be solved by forward substitution. This is because L + D is lower triangular.

Again we can express this method in terms of equations, except in this case we use the components of we already computed. We start off with the same:

To get second vector, we keep 1st component of next vector

This can be programmed easily:

*For k = 1,2,… Number of iterations (N)*

*For I = 1, 2, …, Number of rows (n)*

*End*

*End*

The complexity of the Gauss-Siedel method is whereN = # number of iterations. This is because matrix vector multiplication is

SOR

**SOR** (Successive Over relaxation) = Practical Version of Gauss-Siedel

Typically, you use SOR rather than Gauss-Siedel because you can play with a relaxation parameter to get convergence:

Convergence

As we have seen, each iterative method involves creating a sequence:

Such that if the sequence of vectors converges to , then .

**Convergence Theorem 1:** The sequence converges if and only if the spectral radius of B < 1. This means that the modulus of every eigenvalue of B is less than 1.

Note. The modulus of a complex / real number

Note. The spectral radius of a matrix for = 1, … n

Proof. Consider the sequence of errors

We find that

Note that B has eigenvectors and eigenvectors

Then,

And,

So if you want to determine if an iterative method will work, find eigenvalues of B and see whether spectral radius is less than 1.

BUT… Checking whether the spectral **S(B) < 1,**  is an expensive, perhaps more expensive than solving the original system Ax = b

**Convergence Theorem 2 (Jacobi / Gauss-Siedel):**

If A is diagonal dominant:

Then Jacobi and G-S method converge.

**Convergence Theorem 3 (SOR):** If A is positive-definite, then SOR converges.

Alternative Methods

We can approach finding solution of Ax = b by finding the vector that minimizes

Assuming A is positive definite.

Basically, you take the partial d derivative of g with respect to x, which gives you the gradient, and then travel in the direction of steepest ascent.

* Conjugate Gradient Method
* Stochastic Gradient
* Multigrid Methods / Finite-Element Methods
  + O(n), O(log n), O(1)